Uniquely 2–List Colorable Graphs*

Y. Ganjali G., M. Ghebleh, H. Hajiabolhassan,
M. Mirzazadeh, and B. S. Sadjad

Institute for Studies in Theoretical Physics
and Mathematics (IPM), Tehran, Iran

and

Department of Mathematical Sciences
Sharif University of Technology
P. O. Box 11365–9415, Tehran, Iran

Abstract

A graph is called to be uniquely list colorable, if it admits a list assignment which induces a unique list coloring. We study uniquely list colorable graphs with a restriction on the number of colors used. In this way we generalize a theorem which characterizes uniquely 2–list colorable graphs. We introduce the uniquely list chromatic number of a graph and make a conjecture about it which is a generalization of the well known Brooks' theorem.

1 Introduction

We consider finite, undirected simple graphs. For necessary definitions and notations we refer the reader to standard texts such as [5].

Let G be a graph, $f : V(G) \to \mathbb{N}$ be a given map, and $t \in \mathbb{N}$. An (f, t)-list assignment L to G is a map, which assigns to each vertex v, a set $L(v)$ of size $f(v)$ and $|\bigcup_v L(v)| = t$. By a list coloring for G from such L or an L-coloring for short, we shall mean a proper coloring c in which $c(v)$ is

*The research of second and third authors is supported by the Institute for Studies in Theoretical Physics and Mathematics (IPM), Tehran, Iran.

chosen from $L(v)$, for each vertex v. When $f(v) = k$ for all v, we simply say (k,t)-list assignment for an (f,t)-list assignment. When the parameter t is not of special interest, we say f-list (or k-list) assignment simply. Specially if L is a (t,t)-list assignment to G, then any L-coloring is called a t-coloring for G.

In this paper we study the concept of uniquely list coloring which was introduced by Dinitz and Martin [1] and independently by Mahdian and Mahmoodian [4]. In [1] and [4] uniquely k-list colorable graphs are introduced as graphs who admit a k-list assignment which induces a unique list coloring. In the present work we study uniquely list colorings of graphs in a more general sense.

Definition 1 *Suppose that G is a graph, $f : V(G) \to \mathbb{N}$ is a map, and $t \in \mathbb{N}$. The graph G is called to be* uniquely (f,t)-list colorable *if there exists an (f,t)-list assignment L to G, such that G has a unique L-coloring. We call G to be* uniquely f-list colorable *if it is uniquely (f,t)-list colorable for some t.*

If G is a uniquely (f,t)-list (resp. f-list) colorable graph and $f(v) = k$ for each $v \in V(G)$, we simply say that G is a uniquely (k,t)-list (resp. k-list) colorable graph. In [4] all uniquely 2-list colorable graphs are characterized as follows.

Theorem A [4] *A graph G is not uniquely 2-list colorable, if and only if each of its blocks is either a complete graph, a complete bipartite graph, or a cycle.*

For recent advances in uniquely list colorable graphs we direct the interested reader to [3] and [2].

In developing computer programs for recognition of uniquely k-list colorability of graphs, it is important to restrict the number of colors as much as possible. So if G is a uniquely k-list colorable graph, the minimum number of colors which are sufficient for a k-list assignment to G with a unique list coloring, will be an important parameter for us. Uniquely list colorable graphs are related to defining sets of graph colorings as discussed in [4], and in this application also the number of colors is an important quantity.

In next section we show that for every uniquely 2-list colorable graph G there exists a 2-list assignment L, such that G has a unique L-coloring and there are $\max\{3, \chi(G)\}$ colors used in L.

2 Uniquely $(2, t)$-list colorable graphs

It is easy to see that for each uniquely k-list colorable graph G, and each k-list assignment L to its vertices which induces a unique list coloring, at least $k+1$ colors must be used in L, and on the other hand since G has an L-coloring, at least $\chi(G)$ colors must be used. So the number of colors used is at least $\max\{k+1, \chi(G)\}$ colors. Throughout this section our goal is to prove the following theorem which implies the equality in the case $k = 2$.

Theorem *A graph G is uniquely 2-list colorable if and only if it is uniquely $(2, t)$-list colorable, where $t = \max\{3, \chi(G)\}$.*

To prove the theorem above we consider a counterexample G to the statement with minimum number of vertices. In theorems 4, 6, and 7, we will show that G is 2-connected and triangle-free, and each of its cycles is induced (chordless).

As mentioned above, if G is a uniquely k-list colorable graph, and L a (k, t)-list assignment to G such that G has a unique L-coloring, then $t \geqslant \max\{k+1, \chi(G)\}$. Although the theorem above states that when $k = 2$ there exists an L for which equality holds, this is not the case in general.

To see this, consider a complete tripartite uniquely 3-list colorable graph G. We will call each of the three color classes of G a part. In [3] it is shown that for each $k \geqslant 3$ there exists a complete tripartite uniquely k-list colorable graph. For example one can check that the graph $K_{3,3,3}$ has a unique list coloring from the lists shown in Figure 1 (the color taken by each vertex is underlined).

Suppose that L is a $(3, t)$-list assignment to G which induces a unique list coloring c, and the vertices of a part X of G take on the same color i in c. We introduce a 2-list assignment L' to $G \setminus X$ as follows. For every vertex v in $G \setminus X$, if $i \in L(v)$ then $L'(v) = L(v) \setminus \{i\}$, and otherwise $L'(v) = L(v) \setminus \{j\}$ where $j \in L(v)$ and $j \neq c(v)$. Since L induces a unique list coloring c for G,

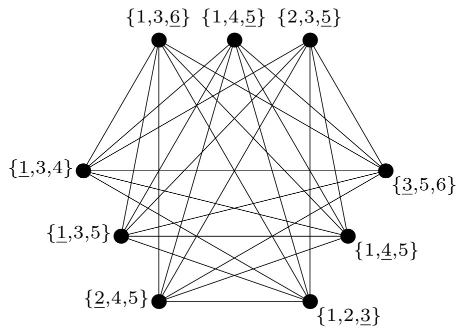

Figure 1: A 3-list assignment to $K_{3,3,3}$ which induces a unique list coloring

$G \setminus X$ has exactly one L'-coloring, namely the restriction of c to $V(G) \setminus X$. But $G \setminus X$ is a complete bipartite graph and this contradicts Theorem A. So on each part of G there must be appeared at least 2 colors and therefore we have $t \geqslant 6$ while $\max\{k+1, \chi(G)\} = 4$.

Similarly one can see that if G is a complete tripartite uniquely k-list colorable graph for some $k \geqslant 3$, and L a (k,t)-list assignment to G which induces a unique list coloring, then on each part there are at least $k-1$ colors appeared and so we have $t \geqslant 3(k-1)$ while $\max\{k+1, \chi(G)\} = k+1$.

Towards our main theorem, we start with two basic lemmas.

Lemma 2 *Suppose that G is a connected graph and $f : V(G) \to \{1,2\}$ such that $f(v_0) = 1$ for some vertex v_0 of G. Then G is a uniquely $(f, \chi(G))$-list colorable graph.*

Proof Consider a spanning tree T in G rooted at v_0 and consider a $\chi(G)$-coloring c for G. Let $L(v)$ be $\{c(v)\}$ if $f(v) = 1$, and $\{c(u), c(v)\}$ if $f(v) = 2$, where u is the parent of v in T. It is easy to see that c is the only L-coloring of G. ■

Lemma 3 *Let G be the union of two graphs G_1 and G_2 which are joined in exactly one vertex v_0. Then G is uniquely $(2,t)$-list colorable if and only if at least one of G_1 and G_2 is uniquely $(2,t)$-list colorable.*

Proof If either G_1 or G_2 is a uniquely $(2,t)$-list colorable graph, by use of Lemma 2 it is obvious that G is also uniquely $(2,t)$-list colorable. On

the other hand suppose that none of G_1 and G_2 is a uniquely $(2,t)$-list colorable graph and L is a $(2,t)$-list assignment to G which induces a list coloring c. Since G_1 and G_2 are not uniquely $(2,t)$-list colorable, each of these has another coloring, say c_1 and c_2 respectively. If $c_1(v_0) = c(v_0)$ or $c_2(v_0) = c(v_0)$ then an L-coloring for G different from c is obtained obviously. Otherwise $c_1(v_0) = c_2(v_0)$, so we obtain a new L-coloring for G, by combining c_1 and c_2. ■

The following theorem is immediate by Lemma 2 and Lemma 3.

Theorem 4 *Suppose that G is a graph and $t \geqslant \chi(G)$. The graph G is uniquely $(2,t)$-list colorable if and only if at least one of its blocks is a uniquely $(2,t)$-list colorable graph.*

Next lemma which is an obvious statement, is useful throughout the paper.

Lemma 5 *Suppose that the independent vertices u and v in a graph G take on different colors in each t-coloring of G. Then the graph G is uniquely (f,t)-list colorable if and only if $G + uv$ is a uniquely (f,t)-list colorable graph.*

The foregoing two theorems are major steps in the proof of Theorem 11. Before we proceed we must recall the definition of a θ-graph. If p, q, and r are positive integers and at most one of them equals 1, by $\theta_{p,q,r}$ we mean a graph which consists of three internally disjoint paths of length p, q, and r which have the same endpoints. For example the graph $\theta_{2,2,4}$ is shown in Figure 2.

Theorem 6 *Suppose that G is a 2-connected graph, $t = \max\{3, \chi(G)\}$, and G is not uniquely $(2,t)$-list colorable. Then G is either a complete or a triangle-free graph.*

Proof Let G be a graph which is not uniquely $(2,t)$-list colorable for $t = \max\{3, \chi(G)\}$, and suppose that G contains a triangle. For every pair of independent vertices of G, say u and v, which take on different colors in each t-coloring of G, we add the edge uv, to obtain a graph G^*. By Lemma 5, G^*

is not a uniquely $(2,t)$-list colorable graph. If G^* is not a complete graph, since it is 2-connected and contains a triangle, it must have an induced $\theta_{1,2,r}$ subgraph, say H (to see this, consider a maximum clique in G^* and a minimum path outside it which joins two vertices of this clique). Suppose that x, y, and z are the vertices of a triangle in H, and $y = v_0, v_1, \ldots, v_{r-1}, v_r = z$ is a path of length r in H not passing through x. Consider a t-coloring c of G^* in which x and v_{r-1} take on the same color. We define a 2-list assignment L to H as follows.

$$L(x) = L(z) = \{c(x), c(z)\}, L(y) = \{c(x), c(y)\},$$

$$L(v_i) = \{c(v_i), c(v_{i-1})\}; \quad \forall\ 1 \leqslant i \leqslant r-1.$$

In each L-coloring of H one of the vertices x and z must take on the color $c(x)$ and the other takes on the color $c(z)$. So y must take on the color $c(y)$ and one can see by induction that each v_i must take on the color $c(v_i)$, and finally x must take on the color $c(x)$. Now since G^* is connected, as in the proof of Lemma 2, one can extend L to a 2-list assignment to G^* such that c is the only L-coloring of G^*. This contradiction implies that G^* is a complete graph, and this means that G has chromatic number $n(G)$, so G must be a complete graph. ∎

Theorem 7 *Let G be a triangle-free 2-connected graph which contains a cycle with a chord and $t = \max\{3, \chi(G)\}$. Then G is uniquely $(2,t)$-list colorable if and only if it is not a complete bipartite graph.*

Proof By Theorem A, a complete bipartite graph is not uniquely 2-list colorable. So if G is uniquely $(2,t)$-list colorable, it is not a complete bipartite graph. For the converse, let G be a graph which is not uniquely $(2,t)$-list colorable where $t = \max\{3, \chi(G)\}$, and suppose that G contains a cycle with a chord. For every pair of independent vertices of G, say u and v, which take on different colors in each t-coloring of G, we add the edge uv, to obtain a graph G^*. By Lemma 5, G^* is not a uniquely $(2,t)$-list colorable graph. If G^* contains a triangle, By Theorem 6, G^* and so G must be complete graphs which contradicts the hypothesis. So suppose that G^* does not contain a triangle.

Consider a cycle $v_1v_2\ldots v_pv_1$ with a chord v_1v_ℓ, and suppose H to be the graph $G^*[v_1, v_2, \ldots, v_p]$. If $v_pv_{\ell-1} \notin E(H)$, there exists a t-coloring c of G^*, such that $c(v_p) = c(v_{\ell-1})$. Assign the list $L(v_i) = \{c(v_i), c(v_{i-1})\}$ to each v_i, where $1 \leqslant i \leqslant p$ and $v_0 = v_p$. Consider an L-coloring c' for H. Starting from v_1 and considering each of two possible colors for it, we conclude that $c'(v_\ell) = c(v_\ell)$. So for each $1 \leqslant i \leqslant p$ we have $c'(v_i) = c(v_i)$. This means that H is a uniquely $(2,t)$-list colorable graph, and similar to the proof of Lemma 2, G^* is a uniquely $(2,t)$-list colorable graph, a contradiction. So $v_pv_{\ell-1} \in E(H)$ and similarly $v_2v_{\ell+1} \in E(H)$. Now consider the cycle $v_1v_2v_{\ell+1}v_\ell v_{\ell-1}v_pv_1$ with chord v_1v_ℓ. By a similar argument, $v_pv_{\ell+1}$ and $v_2v_{\ell-1}$ are in $E(H)$ and so the graph $G^*[v_1v_2v_{\ell+1}v_lv_{\ell-1}v_p]$ is a $K_{3,3}$.

Suppose that K is a maximal complete bipartite subgraph of G^* containing the $K_{3,3}$ determined above. Since G is triangle-free, K is an induced subgraph of G. If $V(G) \setminus V(K) \neq \emptyset$, consider a vertex $v \in V(G) \setminus V(K)$ which is adjacent to a vertex w_1 of K. By 2-connectivity of G^*, there exists a path $vu_1\ldots u_rw_2$ in which $w_2 \in V(K)$ and $u_i \notin V(K)$ for each $0 \leqslant i \leqslant r$. If w_1 and w_2 are in the same part of K, since each part of K has at least 3 vertices, there exists a vertex w_3 other than w_1 and w_2 in the same part of K as w_1 and w_2, and vertices w_1' and w_2' in the other part of K. Considering the cycle $vu_1\ldots u_rw_2w_2'w_3w_1'w_1v$ with chord w_1w_2', by a similar argument as in the previous paragraph, it is implied that v is adjacent to w_3. So v is adjacent to all the vertices of K which are in the same part of K as w_1, except possibly to w_2, but in fact v is adjacent to w_2, since we can now consider w_3 in place of w_2 and do the same as above. This contradicts the maximality of K. On the other hand if w_1 and w_2 are in different parts of K, a similar argument yields a contradiction.

We showed that $G^* = K$ and it is remained only to show that $G = G^*$. If xy is an edge in G^* which is not present in G, using the fact that G is bipartite, one can easily obtain a t-coloring ($t = 3$) of G in which x and y take on the same color, a contradiction. ■

At this point we will consider graphs that do not satisfy the conditions of Theorem 7, namely 2-connected graphs in which every cycle is induced. The following lemma helps us to treat such graphs.

Lemma 8 *A 2-connected graph in which each cycle is chordless, has at least a vertex of degree 2.*

Proof It is a well-known theorem of H. Whitney [6] that a graph is 2-connected, if and only if it admits an ear decomposition (For a description of ear decomposition see Theorem 4.2.7 in [5]). In the case of present lemma, since the graph is chordless, each ear is a path of length at least 2, so the last ear contains a vertex of degree 2. ■

If G is a graph and v a vertex of G, we define G_v to be a graph obtained by identifying v and all of its neighbors to a single vertex $[v]$.

Lemma 9 *If v is a vertex of degree 2 in a graph G, and G_v is uniquely $(2, t)$-list colorable for some t, then G is also uniquely $(2, t)$-list colorable.*

Proof Suppose that v_1 and v_2 are the neighbors of v in G. If L is a $(2, t)$-list assignment to G_v such that G_v has a unique L-coloring, one can assign $L(w)$ to each vertex w of the graph G except v, v_1, and v_2, and $L([v])$ to these three vertices, to obtain a $(2, t)$-list assignment to G from which G has a unique list coloring. ■

The following lemma gives us a family of uniquely $(2, 3)$-list colorable graphs, which we will use in the proof of our main result.

Lemma 10 *Aside from $\theta_{2,2,2} = K_{2,3}$, each graph $\theta_{p,q,r}$ is uniquely $(2, 3)$-list colorable.*

Proof Suppose that $G = \theta_{p,q,r}$ is a counterexample with minimum number of vertices, and u and v are the two vertices of G with degree 3. If one of p, q, and r is 1, then G is a cycle with a chord and we have nothing to prove. Otherwise suppose that one of the numbers p, q, and r, say p is odd, and there exists a vertex w on a path with length p between u and v. Then by Lemma 9 the graph G_w is not a uniquely $(2, 3)$-list colorable graph, a contradiction. Hence $p = 1$ and we yield to the previous case.

So assume that p, q, and r are all even numbers. By the hypothesis at least one of p, q, and r, say r, is greater than 2. If either $p > 2$, $q > 2$, or $r > 4$, by use of Lemma 9 we obtain a smaller counterexample to the

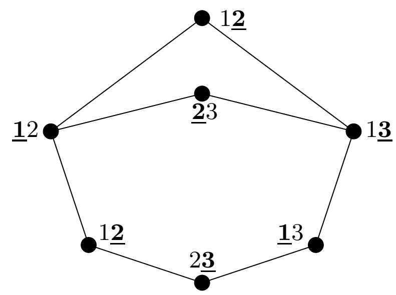

Figure 2: The graph $\theta_{2,2,4}$

statement, which is impossible by minimality of G, so $G = \theta_{2,2,4}$. In Figure 2 there is given a $(2,3)$-list assignment to $\theta_{2,2,4}$ which induces a unique list coloring. This shows that G is a uniquely $(2,3)$-list colorable graph, which contradicts the fact that G is a counterexample to the statement. ■

Now we can prove the main result.

Theorem 11 (MAIN) *A graph G is uniquely 2-list colorable if and only if it is uniquely $(2,t)$-list colorable, where $t = \max\{3, \chi(G)\}$.*

Proof By definition, if G is uniquely $(2,t)$-list colorable for some t, it is uniquely 2-list colorable. So we must only prove that every uniquely 2-list colorable graph G is uniquely $(2,t)$-list colorable for $t = \max\{3, \chi(G)\}$. Suppose that G is a counterexample to the statement with minimum number of vertices. By Theorem 4, G is 2-connected, by Theorem 6, it is triangle-free (by Theorem A it can not be a complete graph), and by Theorem 7, it does not have a cycle with a chord, so Lemma 8 implies that G has a vertex v with exactly two neighbors v_1 and v_2.

Consider the graph $H = G \setminus v$ and note that since $\deg v = 2$, we have $\max\{3, \chi(H)\} = \max\{3, \chi(G)\}$. So if H is uniquely 2-list colorable, by minimality of G, the graph H must be uniquely $(2,t)$-list colorable, and since $t \geqslant 3$ and $\deg v = 2$, we conclude that G is uniquely $(2,t)$-list colorable, a contradiction. Therefore H is not a uniquely 2-list colorable graph and because it is a triangle-free graph, by Theorem A every block of H is either a cycle of length at least four or a complete bipartite graph. This shows that $t = 3$.

We will show by case analysis that G has an induced subgraph G' which is isomorphic to some $\theta_{p,q,r} \neq \theta_{2,2,2}$ (except in the case (i.2)). The graph G' is uniquely $(2,t)$-list colorable by Lemma 10. Now a $(2,3)$-list assignment to G' with a unique list coloring can simply be extended to the whole of G. This completes the proof. To show the existence of G' we consider two cases.

(i) The graph H is 2-connected. So H is either a K_2, a cycle, or a complete bipartite graph with at least two vertices in each part. If $H = K_2$ then $G = K_3$, a contradiction.

  (i.1) If H is a cycle, G is a θ-graph and $G' = G$. Note that since G is not uniquely 2-list colorable, $G' = G$ is not isomorphic to $\theta_{2,2,2}$.

  (i.2) If H is a complete bipartite graph, since G is triangle-free, v_1 and v_2 are in the same part in H. Now there must exist at least one other vertex v_3 in that part –otherwise G will be a complete bipartite graph. Suppose that u_1 and u_2 are two vertices in the other part of H. The graph G' induced from G on $\{v, v_1, v_2, v_3, u_1, u_2\}$ is a uniquely $(2,3)$-list colorable with the list assignment L as follows: $L(v) = \{1,2\}$, $L(v_1) = \{1,3\}$, $L(v_2) = \{1,2\}$, $L(v_3) = \{2,3\}$, $L(u_1) = \{2,3\}$, $L(u_2) = \{1,3\}$.

(ii) The graph H is not 2-connected. Since G is 2-connected H has exactly two end-blocks each of them contains one of v_1 and v_2.

If all of the blocks of H are isomorphic to K_2, then G is a cycle which is impossible. So H has a block B with at least three vertices. Since B is a cycle or a complete bipartite graph with at least two vertices in each part, it has an induced cycle C which shares a vertex with at least two other blocks. Since G is 2-connected these two vertices must be connected by a path disjoint from B. Suppose that P is such a path with minimum length. The graph $G' = C \cup P$ is the required θ-graph. ■

3 Concluding remarks

We begin with a definition which is a natural consequence of the aforementioned results.

Definition 12 *For a graph G and a positive integer k, we define $\chi_u(G,k)$ to be the minimum number t, such that G is a uniquely (k,t)-list colorable graph, and zero if G is not a uniquely k-list colorable graph. The* uniquely list chromatic number *of a graph G, denoted by $\chi_u(G)$, is defined to be $\max_{k\geqslant 1}\chi_u(G,k)$.*

In fact Theorem 11 states that for every uniquely 2-list colorable graph G, $\chi_u(G,2) = \max\{3,\chi(G)\}$ and by Brooks' theorem and the fact that for every uniquely 2-list colorable graph G, $\Delta(G) \geqslant 3$, we have shown that $\chi_u(G,2) \leqslant \Delta(G)+1$. This seems to remain true if we substitute 2 by any positive integer k.

Conjecture 13 *For every graph G we have $\chi_u(G) \leqslant \Delta(G)+1$, and equality holds if and only if G is either a complete graph or an odd cycle.*

The above conjecture implies the well-known Brooks' theorem, since for every graph G we have $\chi_u(G,1) = \chi(G)$, and so $\chi(G) \leqslant \chi_u(G)$. Hence the above conjecture implies that $\chi(G) \leqslant \Delta(G)+1$. On the other hand if $\chi(G) = \Delta(G)+1$, we will have $\chi_u(G) = \Delta(G)+1$ and the conjecture above implies that G is either a complete graph or an odd cycle.

Acknowledgements

The authors are grateful to Professor E. S. Mahmoodian for his comments and support. We also thank the anonymous referees for their inquiry and useful comments.

References

[1] J. H. Dinitz and W. J. Martin. The stipulation polynomial of a uniquely list-colorable graph. *Austral. J. Combin.*, 11:105-115, 1995.

[2] Ch. Eslahchi, M. Ghebleh, and H. Hajiabolhassan. Some concepts in list coloring. (submitted).

[3] M. Ghebleh and E. S. Mahmoodian. On uniquely list colorable graphs. (submitted).

[4] M. Mahdian and E. S. Mahmoodian. A characterization of uniquely 2-list colorable graphs. *Ars Combin.* (to appear).

[5] D. B. West. *Introduction to graph theory.* Prentice Hall, Upper Saddle River. NJ, 1996.

[6] H. Whitney. Congruent graphs and the connectivity of graphs. *Amer. J. Math.*, 54:150-168, 1932.